JOURNAL OF APPROXIMATION THEORY **62**, 274–276 (1990)

Note

A Problem on Approximation by Fourier Sums with Monotone Coefficients

SONG PING ZHOU*

Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, People's Republic of China

Communicated by V. Totik

Received June 13, 1988; revised September 14, 1988

Let $C_{2\pi}$ be the class of real, continuous functions with period 2π. $E_n^*(f)$ is the best approximation to $f(x) \in C_{2\pi}$ by trigonometric polynomials of degree $\leqslant n$, and $S_n(f, x)$ is the nth partial sum of the Fourier series of $f(x)$. Write

$$\|f\| = \max_{-\pi \leqslant x \leqslant \pi} |f(x)|.$$

It is well known that

$$\|f - S_n(f)\| = O(\log(n+1)\, E_n^*(f)), \tag{1}$$

and, in general, the factor $\log(n+1)$ in (1) cannot be improved. However, one may hope that, for $f(x)$ in some subclass of $C_{2\pi}$, a better estimate holds, e.g.,

$$\|f - S_n(f)\| = O(E_n^*(f)). \tag{2}$$

Recently, Professor Tingfan Xie [1] asked

Problem 1. Does there exist a positive constant M such that, for every function $f \in C_{2\pi}$ with positive Fourier coefficients,

$$\|f - S_n(f)\| \leqslant M E_n^*(f) \qquad (n = 1, 2, ...)?$$

Later, in a seminar, he asked

Problem 2. Does there exist a positive constant M such that for every function $f \in C_{2\pi}$ with monotone Fourier coefficients,

$$\|f - S_n(f)\| \leqslant M E_n^*(f) \qquad (n = 1, 2, ...)?$$

* Present address: Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, NS, Canada B3H 3J5.

0021-9045/90 $3.00

Some time ago, the author constructed an example showing that the answer to Problem 1 is negative. But the corresponding Fourier series was lacunary, so it could not apply to Problem 2. In the present paper we give a carefully constructed counterexample which indicates that the answer to Problem 2 is still negative.

THEOREM. *There exists a function $f(x) \in C_{2\pi}$ with strongly monotone Fourier coefficients such that*

$$\overline{\lim_{n \to \infty}} \frac{\|f - S_n(f)\|}{(\log n)\, E_n^*(f)} > 0.$$

A strongly monotone sequence φ_n is one for which all $\varphi_n > 0$ and $n\varphi_n$ decreases. It is evident that if φ_n is strongly monotone, then it decreases.

Proof of the Theorem. Let

$$a_n = 2^{-k^2} j^{-1} \qquad \text{for} \quad 2_k + 1 \leqslant n = 2^k + j \leqslant 2^{k+1},\ k = 0, 1, 2, ...,\ \ n = 1, 2, ...$$

Obviously $a_n \geqslant 0$ for all n. Furthermore,

$$na_n = \frac{2^k + j}{2^{k^2} j} > \frac{2^k + j + 1}{2^{k^2}(j+1)} = (n+1)a_{n+1}$$

for $j = 1, 2, ..., 2^k - 1$, $k = 1, 2, ...,$ and

$$(2^{k+1} + 1)a_{2^{k+1}+1} = \frac{2^{k+1} + 1}{2^{(k+1)^2}} < \frac{2^{k+1}}{2^{k^2+k}2^k} \leqslant 2^{k+1} a_{2^{k+1}} \qquad \text{for} \quad k = 0, 1, 2, ...;$$

hence, na_n decreases.

Now define

$$f(x) = \sum_{n=2}^{\infty} a_n \cos nx = \sum_{k=0}^{\infty} \frac{1}{2^{k^2}} \sum_{j=1}^{2^k} \frac{\cos(2^k + j)x}{j}.$$

It is not difficult to see that

$$\|f - S_{2^k}(f)\| = f(0) - S_{2^k}(f, 0) = \frac{1}{2^{k^2}} \sum_{j=1}^{2^k} \frac{1}{j} + O\left(\frac{1}{2^{k^2}}\right)$$

$$= \frac{k}{2^{k^2}} + O\left(\frac{1}{2^{k^2}}\right).$$

On the other hand

$$\left\| f(x) - S_{2^k}(f, x) - \frac{1}{2^{k^2}} \sum_{j=1}^{2^k} \frac{\cos(2^k + j)x}{j} \right\| \leqslant \frac{\|\sin 2^n x\|}{2^{k^2 - 1}} \left\| \sum_{j=1}^{2^k} \frac{\sin jx}{j} \right\|.$$

By the well-known inequality

$$\left|\sum_{j=1}^{m} \frac{\sin jx}{j}\right| \leqslant 3\sqrt{\pi} \qquad \text{for all } m,$$

it follows that

$$E_{2^k}^*(f) = O\left(\frac{1}{2^{k^2}}\right).$$

Therefore

$$\overline{\lim_{n \to \infty}} \frac{\|f - S_n(f)\|}{(\log n)\, E_n^*(f)} > 0.$$

Thus the theorem is proved.

Acknowledgments

The author is very grateful to Professor Tingfan Xie for his instructions, direction, and, especially, for his simplification of the proof of the Theorem. Many thanks are also due to Professor V. Totik for his valuable comments.

Reference

1. Tingfan Xie, Problem No. 1, *Approx. Theory Appl.* **3** (1987), 144.